Regulæ variabiliter differentiandi generales,

quas

Consent. Ampliss. Ord. Philosoph.,

Præside

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Quo quidem valore admisso, erit $\varphi x + \varphi y = CL(1+cx)(1+cy) + 2A$, atque $\varphi(x+y+cxy) = CL(1+c(x+y+cxy)) + A = CL(1+cx)(1+cy) + A$, quæ igitur conveniunt, si const. $A = o$.

Similiter si $(xy) = \frac{x+y+axy}{1-cxy} = z$, *erit reversim* $y = \frac{z-x}{1+ax+cxz}$,

ideoque si $z = x + rdx$, $y = \frac{rdx}{1+ax+cx(x+rdx)} = \frac{rdx}{1+ax+cx^2} + X_2 r^2 dx^2 + \ldots$,

ideoque $\varphi_1 x = X_1 . C = \frac{C}{1+ax+cx^2}$, si $\varphi\left(\frac{x+y+axy}{1-cxy}\right) = \varphi x + \varphi y$, eritque $\varphi x = \text{L} x$, si $a = o$ & $c = 1$, alias vero per Log. vel L exprimetur *). Quia

scilicet $d\,\text{L}\,x = \frac{dx}{1+x^2}$, erit $d\text{L}\left(\frac{\alpha+\beta x}{\gamma+\delta x}\right) = \frac{d\left(\frac{\alpha+\beta x}{\gamma+\delta x}\right)}{1+\left(\frac{\alpha+\beta x}{\gamma+\delta x}\right)^2} = \frac{(\beta\gamma - \alpha\delta).dx}{\overline{\gamma+\delta x}^2 + \overline{\alpha+\beta x}^2}$

quod facile $= \frac{Cdx}{1+ax+cx^2}$ efficias, ideoque hæc $\varphi x = \text{L}\frac{\alpha+\beta x}{\gamma+\delta x} +$ Const. erit. Sic si $c = 1$ & $a = 2C\varphi$, faciendum erit $\gamma^2 + \alpha^2 : 2(\gamma\delta + \alpha\beta) : \delta^2 + \beta^2 = 1 : 2C\varphi : 1$ seu $\gamma^2 + \alpha^2 = \beta^2 + \delta^2$ atque $(\alpha^2 + \gamma^2)C\varphi = \alpha\beta + \gamma\delta$, unde duæ quantitatum α, β, γ & δ determinantur. Quoniam igitur duæ arbitrio relinquuntur, faciamus $\alpha = o$, eritque $\delta = \gamma C\varphi$, atque $\beta^2 = \gamma^2(1 - C\varphi^2)$, seu $\beta = \gamma\, S\varphi$, quare jam $\varphi x = \text{L}\frac{xS\varphi}{1+xC\varphi} +$ Const. Facile vero negotio hunc const. evanescentem invenies. Hæc vero functio in Dilogarithmicis occurrit. (v. Spec. Exc. Anal. p. 56).

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*) Conf. intr. in elem. funct. Ellipt. Theor. p. 72. sq. & Theor. funct. interat. passim.

Hic vero modus differentiandi ad omnes functiones secundum talem formulam $\varphi x + \varphi y = \varphi z$ addibiles quadrat, quales Trigonometricæ & Ellipticæ sunt, plurimæque ex arte *iterandi oriundæ* hujus sunt indolis.

5) Ex hac dicta arte, quæ plurium est origo, variæ differentiandi regulæ colliguntur, quas, dum ipsam tractaturi sumus, exponemus. Universim vero originem functionum pressius spectes, ut si imaginarii consideratio vel evolutio secundum aliud argumentum functionem derivandam genuit, ex harum theoria rem absolvas.

Postquam vero jam calculi differentialis præcipua quædam momenta & capita, quæ suam quæque propriam merentur tractationem, breviter indicavimus, ad id, unde egressi fuimus revertamur. Ostendimus scil. evolutionem functionis polynomii $z + rz_1 + r^2z_2 + r^3z_3 + ..$ vel ex Theor. Tayloreo & binomiali sæpius adhibito peti, vel ad *differentationem variabilem* revocari, dummodo $dz = z_1$, $d(dz) = d^2z$, $d(d^2z) = d^3z$ &c. $\underset{c}{d^2}z = \frac{d^2z}{1.2} = z_2$, $\underset{c}{d^3}z = \frac{d^3z}{1.2.3} = z_3$ &c. facimus; seu, ut verius dicamus, functionem functionis ita evolvi, quod tamen eodem redit. Si enim quadrinomium modo descriptum adsumimus, hoc ut talem ipsius x functio $\mathfrak{z}^1x$ considerare licet, ut formula polynomialis evolvenda $\varphi(z + rz_1 + r^2z_2 + r^3z_3)$ ad hanc binomialem $\varphi(\mathfrak{z}^1(x+r))$ redeat. Faciamus enim $z = \mathfrak{z}^1x = \mathfrak{z}_0 + \mathfrak{z}_1x + \mathfrak{z}_2x^2 + \mathfrak{z}_3x^3$, eritque $\frac{dz}{dx} = \mathfrak{z}_1 + 2.\mathfrak{z}_2x + 3.\mathfrak{z}_3x^2$, quod $= \mathfrak{z}_1^1x$ (seu 2^1x) ponamus, rursumque $\frac{d^2z}{(dx)^2} = 2.(\mathfrak{z}_2 + 3.\mathfrak{z}_3x) = 2.\mathfrak{z}_2^1x (= 2.1^1x)$ atque $d^3z = 2.3.\mathfrak{z}_3.dx^3$. Jam vero numeri $\mathfrak{z}_0$, $\mathfrak{z}_1$, $\mathfrak{z}_2$, $\mathfrak{z}_3$, ita sunt determinandi, ut $z + rz_1 + r^2z_2 + r^3z_3 = \mathfrak{z}^1(x+r)$ fiat, quod evolutum valet $\mathfrak{z}^1x + \mathfrak{z}_1^1x.r + \mathfrak{z}_2^1.r^2 + \mathfrak{z}_3.r^3$, ideoque erit non modo $\mathfrak{z}^1x = z$,

ut posuimus, sed etiam $\mathfrak{z}^1_1 x = z_1$ seu $z_1 = \frac{dz}{dx}$, $z_2 = \frac{1}{2}\cdot\frac{d^2z}{dx^2}$, atque $z_3 = \mathfrak{z}_3 =$ $\frac{d^3z}{2.3.dx^3} = \underset{c}{d}^3 z : dx^3$, quod quidem dictis valoribus convenit, si $dx = 1$ facimus. Patet vero jam, x & r, sicuti z, z_1, z_2 & z_3, quibusvis exsistentibus, hinc $\mathfrak{z}_0$, $\mathfrak{z}_1$, $\mathfrak{z}_2$ & $\mathfrak{z}_3$ determinatum iri, idque absque contradictione.

Ut vero hujus maximæ rei exemplum facile exhibeamus, sit evolvendum $(z + z_1 + z_2)^n$ vel $\overline{z + rz_1 + r^2 z_2}^{\,n}$ (ubi postmodum $r = 1$ facias, si placet): erit quidem ex regulis algebraicis hoc

$$= z^n + nz^{n-1}(rz_1 + r^2 z_2) + n_2 z^{n-2}(rz_1 + r^2 z_2)^2 + \ldots\, ;$$

ex his vero analyticis faciendum est $z_1 = dz$, & $z_2 = \frac{d^2z}{2}$, atque

$(z + rz_1 + r^2 z_2)^n = z^n + r.d\widehat{z^n} + \frac{r^2 d^2(z^n)}{1.2} + \frac{r^3. d^3(z^n)}{1.2.3} + \ldots$ erit. Est scilicet

$$d(z^n) = nz^{n-1}dz,\quad d^2(z^n) = n.d(z^{n-1}dz) = n.z^{n-1}d^2z + n\overline{n-1}z^{n-2}\widehat{dz}^2,$$

$$d^3(z^n) = \left\{\begin{array}{l} nd\overline{z^{n-1}d^2z} + \\ + n\overline{n-1}\,d\,\overline{z^{n-2}dz^2} \end{array}\right\} = \left\{\begin{array}{l} nz^{n-1}d^3z + n.\overline{n-1}z^{n-2}dzd^2z \\ + n\,\overline{n-1}\,\overline{n-2}\,z^{n-3}\overline{dz}^3 + n.\,\widehat{n-1}.2z^{n-2}dzd^2z \end{array}\right\}$$

seu $d^3(z^n) = nz^{n-1}d^3z + 3n\overline{n-1}z^{n-2}dzd^2z + n\overline{n-1}\,\overline{n-2}z^{n-3}d^3z$, &c.

Unde jam $(z + rdz + r^2 \underset{c}{d}^2 z)^n = z^n + r.nz^{n-1}dz + \frac{r^2}{1.2}\cdot(nz^{n-1}d^2z + n\overline{n-1}z^{n-2}\widehat{dz}^2) +$

$+\ \frac{r^3}{1.2.3}\cdot(nz^{n-1}d^3z + 3n\overline{n-1}z^{n-2}dzd^2z + n\,\overline{n-1}\,\overline{n-2}z^{n-3}dz^3) + \ldots$

seu, si jam z_1 pro dz & $2z_2$ pro d^2z &c. posueris, $(z + rz_1 + r^2 z_2)^n =$ $= z^n + nrz^{n-1}z_1 + r^2(nz^{n-1}z_2 + n_2.z^{n-2}z_1^{\,2}) + r^3.\,(nz^{n-1}z_3 + 2n_2 z^{n-2}z_1 z_2 + n_3.z^{n-3}z_1^{\,3}) + \ldots$ quod quidem valori algebraico modo exhibito fusiusque explicato convenire judices.

Sed jam universim, ut ostendimus, erit $(\varphi z^1=)\varphi(z+rdz+r^2\underset{c}{d}^2z+r^3\underset{c}{d}^3z+\ldots)$ $=\varphi z+rd\varphi z+r^2\underset{c}{d}^2\varphi z+\ldots$ seu $\varphi(z+rdz+\frac{r^2d^2z}{1.2}+\frac{r^3d^3z}{1.2.3}+\ldots)=\varphi z+r.d\varphi z+\ldots$ $+\frac{r^2d^2\varphi z}{1.2}+\frac{r^3d^3\varphi z}{1.2.3}+\ldots$, ideoque functio polynomii (vel etiam functio functionis, si $\left\{\begin{matrix} z=fx \\ \& \;\; z^1=\overline{f(x+r)} \end{matrix}\right\}$ facis) variabiliter differentiando evolvitur, sicuti inverse differentialia altiora cum argumento variabile ad talem evolutionem revocantur: Id quod *Theorema polynomiale* constituit.

Quoniam vero differentiatio plerumque simplicissima habetur, evolutio illa facillime inde institui, & hæc ita omnibus numeris absoluta videtur. Superest quidem adhuc alia multo latius patens evolutio, ea scilicet, qua exigitur, ut functio polynomiorum plurium $x+r\underset{c}{d}x+r^2d^2x+\ldots, y+rdy+r^2\underset{c}{d}^2y+\ldots,$ $z+rdz+r^2\underset{c}{d}^2z+$ &c. secundum r evolvatur, sed et hæc ad illam revocabis, si attenderis, hanc ut functionem ipsius r considerari posse, ideoque hanc quæstionem eo redire, ut evolvatur functio quædam ipsius r secundum potentias integras ipsius r: quod quidem ex ipso Theoremate Tayloreo petiit *Maclaurin*. Quoniam enim $f(x+r)=fx+rf_1x+r^2f_2x+\ldots$, erit, faciendo $x=o$, $fr=fo+rf_1o+r^2f_2o+r^3f_3o+\ldots$, quæ quidem formula huic quidem casui semper applicari poterit, cum x, y, z horumque differentialia indeterminata reliquantur. Nihilo tamen minus peculiarem hæc evolutio sibi arrogat expositionem.

Attamen ipsa dicta differentiatio aliquid negotii superflui facessit. Sic jam ad $d^3(x^n)$ in unum colligendi erant termini, factore $dz\ d^2z$ gaudentes. Sic quoniam $dfx=dx\underset{x}{\partial}fx$, seu brevius $df=dx\partial f$, indicante scilicet ∂f primam ipsius f functionem derivatam (Ex. gr. $\partial(x^n)=nx^{n-1}$), iterum iterumque differentiando obtinebitur

$$d^2fx = \overbrace{dx^2}.\underset{x}{\partial^2} fx + d^2x.\underset{x}{\partial} fx\,,$$

$$d^3fx = \overbrace{dx^3}.\underset{x}{\partial^3} fx + 3dx\,d^2x\underset{x}{\partial^2} fx + d^3x\underset{x}{\partial} fx$$

$$d^4fx = \overbrace{dx^4}\partial^4 fx + 6\widehat{dx^2}d^2x\partial^3 fx + (3\widehat{d^2x^2} + 4dx\,d^3x)\partial^2 fx + d^4x\partial fx, \text{ \&c.}$$

Ubi jam hoc accidit, ut d^4f non nisi ex V partibus constet, cum tamen ex differentiatione ipsius d^3f immediate VII (nempe $= 2+3+2$, sicuti ex numero factorum partium in d^3f constat) partes obtineantur, quarum igitur II cum reliquis in unum collecta sunt. Jam vero si d^5f ex differentiatione ipsius d^4f quæreremus, immediate ita partes $(2+3+2+3+2=)$XII obtineremus, quæ tamen in unum collectæ non nisi VII diversas, numeris coefficientibus præditas, constituunt. In altioribus vero multo major erit reductio: jam scilicet complete hoc d^5f differentiando, obtinentur XIX termini, qui in XI contrahuntur &c. Haud igitur exigui res est momenti, si differentiale altius absque aliqua ejusmodi reductione obtineri possit: id quod ex his *regulis* conficitur:

1) *Differentiale* (d^nf) *ordinis cujuscunque* (n) *jam explicito, differentiale* ($d^{n+1}f$) *ordinis mox subsequentis* ($n+1$) *inde elicietur, differentiale istud explicitum* (d^nf) *in numerum ordinis sequentis* ($n+1$) *ducendo, terminumque illius expliciti quemvis non nisi secundum duo sua altissima sibique invicem consequentia ipsius x differentialia (inferiobus scilicet, quæ huic termino insunt, omnibus pro constantibus habitis) differentiando, deindeque hunc terminum tum per interiorem tum per exteriorem differentialis ipsius x, quod jam adest, altissimi exponentem dividendo.*

In $(d^rx)^m$ vero nominamus *m exteriorem* seu gradus, *r* vero *interiorem* seu ordinis exponentem.

1) Ubi tamen observandum, ipsum x (utut scilicet $=d^0x$) et dx, sicuti dx & d^2x, itemque generatim d^rx & $d^{r+1}x$, sibi invicem consequi; ideoque in d^nf terminum $(dx)^n.\underset{x}{\partial^n}fx$ semper ex regula jam proposita hos duos $(dx)^{n+1}\underset{x}{\partial^{n+1}}fx+\frac{\overline{n+1}\,n}{2}\overline{dx}^{n-1}d^2x.\underset{x}{\partial^n}fx$ in $d^{n+1}f$ suppeditare; utrumque enim, et x et dx, in hoc termino varientur necesse est, ideoque simpliciter secundum x et dx differentiando obtinetur

$$(dx)^{n+1}\partial^{n+1}f+n(dx)^{n-1}d^2x\partial^nf\ ,$$

deindeque ex regula utrumqua terminum in $(n+1)$ ducendo, illumque per $1.\overline{n+1}$, huncque per 2.1 seu per utrumque utriusque exponentem altissimi differentialis (sc. ipsius $\overline{d^1x}^{n+1}$ in altero atque ipsius $(d^2x)^1$ in altero termino) dividendo, termini, quos diximus, habebuntur.

2) Sin vero Ex. gr. ex termino $3dxd^2x\partial^2f$ in d^3f terminos, qui exinde in d^4f exorientur, quærimus; quoniam dx atque d^2x altissima in isto duo sunt differentialia, eaque sibi invicem proxime consequentia, secundum hæc ambo differentiando (ipsoque x ideoque et ∂^2f seu ∂^2fx pro constante habito), obtinemus primum $3((d^2x)^2+dxd^3x)\partial^2f$ seu duos terminos mediatrices, deindeque utrumque horum in 4,(d^4f enim quæritur), ducendo, illumque per 2.2 (exponentes ipsius $(d^2x)^2$), huncque per 3.1 (seu exponentes ipsius $(d^3x)^1$) dividendo, habebimus $(3(d^2x)^2+4dxd^3x)\partial^2f$ pro parte desiderata in d^4f.

3) Tandem observandum est, si in termino quodam ipsius d^nf differentialia adfuerint dx, $d^2x\ldots\ldots d^{r-2}x, d^rx$, deficiente $d^{r-1}x$, tum *solum* d^rx *pro variabile* esse habendum, cetera vero $x, dx, d^2x\ldots\ldots d^{r-2}x$ pro constantibus: altissima enim etsi $d^{r-2}x$ atque d^rx sunt, tamen hæc sibi invicem non consequuntur, cum inter ipsa medium sit $d^{r-1}x$, ideoque non nisi secun-

dum $d^r x$ differentiandum esse patet. Sic Ex. gr. terminus $d^3x.\partial f$ in $d^3 f$ suppeditat tantummodo $d^4x\partial f$ in $d^4 f$ (factor enim est $\frac{4}{4.1}=1$).

4) Si igitur in exemplo primò sumseris $n=3$; omniaque quæ in his exemplis (1, 2, 3) obtinuimus, in unum collegeris, habebis d^4fx et quidem isti (d^4f), quod directa differentiatione obtinuimus, omnino conveniens.

5) Si vero jam pro ultimo exemplo proponimus d^5f ex d^4f inveniendum, primum differentiatione secundùm differentialia in unoquoque ipsius d^4f termino altissima & sibi invicem consequentia, ex regula instituta, habebimus hos terminos mediatrices:

$$\widehat{dx}^5\partial^5 f+4.\widehat{dx}^3d^2x\partial^4 f+6(2dx\widehat{d^2x}^2+\widehat{dx}^2d^3x)\partial^3 f+(3.2d^2xd^3x+4dxd^4x)\partial^2 f+d^5x\,\partial f,$$

deindeque patet, horum numeros coefficientes suo ordine sepositos: 1, 4, 6.2, 6, 3.2, 4 & 1, singulos ex regula præscripta in $\frac{5}{1.5}$, $\frac{5}{2.1}$, $\frac{5}{2.2}$, $\frac{5}{3.1}$, $\frac{5}{3.1}$, $\frac{5}{4.1}$, $\frac{5}{5.1}$ respective esse ducendos, ideoque in 1, 10, 15, 10, 10, 5, 1, mutandos; quare totum

$$d^5f=\widehat{dx}^5.\partial^5 f+10\widehat{dx}^3d^2x\partial^4 f+(15dx\widehat{d^2x}^2+10\widehat{dx}^2d^3x)\partial^3 f+$$
$$+(10d^2xd^3x+5dxd^4x)\partial^2 f+d^5x.\partial f$$ evadit; id, quod, completam instituendo differentiationem, omnia (x, dx, d^2x, d^3x & d^4x) scilicet variando atque postmodum ordinando & in unum colligendo, comprobare potes. Et similiter ex eadem regula pedetentim operando d^6f, d^7f etc. immediate sub forma quam maxime contracta obtinebis.

Hic vero noster operandi modus ita breviter indicari potest: $d^5f=5.\overset{1}{d}(d^4f)$, seu generatim $d^nf=n.\overset{1}{\underset{c}{d}}(d^{n-1}f)$; indicante scilicet $\overset{1}{\underset{\bullet}{d}}$ *differentiationem* quidem, (quare $\overset{1}{\underset{c}{d}}(p+q)=\overset{1}{\underset{c}{d}}p+\overset{1}{\underset{c}{d}}q$ atque $\overset{1}{\underset{\bullet}{d}}\widehat{pq}=q\overset{1}{\underset{c}{d}}p+p\overset{1}{\underset{c}{d}}q$), *singularem* tamen, nempe ejusmodi, ut *in termino quocunque* (p vel q) *non*

nisi altissima, quæ in hoc adsunt, sibique invicem immediate consequuntur, differentialia, Ex. gr. $d^{r-1}x$ & $d^r x$, *varientur, reliquis pro constantibus habitis, prætereaque, differentiatione ita rite peracta, novorum quisque terminus per utrosque, et exteriorem, et interiorem, altissimi, quod jam obtinebatur, differentialis exponentem dividatur.* Scilicet significat, si r in suo termino maximi est ordinis exponens,

$$\overset{1}{\underset{c}{d}}(F(d^{r-1}x).(d^r x)^s) = F(d^{r-1}x).s(d^r x)^{s-1}.\frac{d^{r+1}x}{(r+1).1} + F_1(d^{r-1}x).\frac{(d^r x)^{s+1}}{r.(s+1)},$$

exsistente $F_1 u = \frac{dFu}{du}$ atque $F(d^{r-1}x)$ functione ipsorum $x, dx, d^2x \ldots d^{r-1}x$,

quorum tamen nullum præter $d^{r-1}x$, jubente $\overset{1}{\underset{c}{d}}$, variat. Patet vero, excepto casu $r=1$, quem jam antea tractavimus, et pro quo in signo admisso valet $\overset{1}{\underset{c}{d}}(Fx.(dx)^s) = Fx.\frac{s}{2}(dx)^{s-1}d^2x + F_1 x.\frac{(dx)^{s+1}}{s+1}$, $F(d^{r-1}x)$ sub forma $A_{r-2}\,(d^{r-1}x)^m$ semper haberi, exsistente A_{r-2} functione ipsorum $x, dx, d^2x \ldots d^{r-2}x$ algebraica aliqua, eaque ideo & integra & ab $\overset{1}{\underset{c}{d}}$ nullo modo affecta: quare

$$\overset{1}{\underset{c}{d}}(A_{r-2}.\overline{d^{r-1}x}^m.\widehat{d^r x}^s) = A_{r-2}.\overset{1}{\underset{c}{d}}(\overline{d^{r-1}x}^m.\widehat{d^r x}^s) = A_{r-2}\, m\overline{d^{r-1}x}^{m-1}.\frac{\widehat{d^r x}^{s+1}}{r.(s+1)} +$$

$$+ A_{r-2}.s.\overline{d^{r-1}x}^m.\widehat{d^r x}^{s-1}.\frac{d^{r+1}x}{(r+1).1}, \text{ seu } \overset{1}{\underset{c}{d}}(A_{r-2}\overline{d^{r-1}x}^m\,\widehat{d^r x}^s) =$$

$$A_{r-2}\left(\frac{m}{r.\overline{s+1}}.\overline{d^{r-1}x}^{m-1}\,\widehat{d^r x}^{s+1} + \frac{s}{r+1}.\,\overline{d^{r-1}x}^m\,\widehat{d^r x}^{s-1}\,d^{r+1}x\right).$$

Sumto vero $m=o$, quo quidem casu $d^{r-1}x$ ab isto termino abest, obtinetur $d\,(A_{r-2}.\,\widehat{d^r x}^s) = \frac{s}{r+1}A_{r-2}.\,\widehat{d^r x}^{s-1}d^{r+1}x$. Si igitur in termino quodam exponentes interiores differentialium, quæ in isto adsunt, altissimorum ultra unitatem differunt (ut ex. gr. $d^r x$ & $d^{r-2}x$ vel $d^{r-3}x$ &c.), solum differentiale

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altissimum ($d^r x$) variandum est, et quidem singulari, quem docuimus, modo (per $\overset{1}{\underset{c}{d}}$ indicato) differentiando. Hæ vero formulæ regulas jam præscriptas breviter exprimunt. Patet vero, calculatorem excercitatum differentiationem, multiplicationem & divisionem, si has regulas aliquando bene perspexerit, junctim seu uno actu perficere posse, ut ita differentiale quæsitum currente calamo perscribat. Præterea observamus, in terminis extremis ($dx^n \partial^n f$ et $d^n x \partial f$) coefficientes unitatem æquare, ideoque in his efformandis multiplicationem et divisionem, a regula præscriptas, omnino omitti posse; reliquosque semper fore numeros integros.

7) Ex regulis jam traditis $d^n f$ ab $d^{n-1} f$, ideoque ut dicunt *recurrenter*, derivatur, at utrumque absolute, (seu a se invicem *independenter*, ut dicunt) describendi ars quæri potest.

Jam vero primum ipsius $d^n f$ terminum $(dx)^n \partial^n f$ obtinuimus, indeque ipsius $d^{n+1} f$ duo priores $\widehat{dx}^{n+1} . \partial^{n+1} f + \frac{n.\overline{n+1}}{2} \widehat{dx}^{n-1} d^2 x \partial^n f + \ldots\ldots$ ex regula præscripta facile elicies: tum igitur inde, tum ex ipsis exemplis jam allatis, colligitur, hanc ipsius $d^n f$ fore formam:

$$d^n f = X_0 \partial^n f + X_1 \partial^{n-1} f + X_2 \partial^{n-2} f + X_3 \partial^{n-3} f + \ldots.$$

ubi X_0, X_1, X_2, et reliqui coefficientes per dx, $d^2 x$, $d^3 x$ etc. exprimuntur; at quomodo? Ex modo dictis quidem colligitur $X_0 = \widehat{dx}^n$, atque $X_1 = \frac{n.\overline{n-1}}{2} \widehat{dx}^{n-2} d^2 x$, seu $X_1 = \overset{1}{\underset{c}{d}}(n \widehat{dx}^{n-1})$, vel etiam $X_1 = \underset{c}{d}(\underset{1}{\partial} \widehat{dx}^n) = \underset{c}{d} \underset{1}{\partial} X_0$, indicante $\underset{1}{\partial}$ derivationem secundum dx; X_1 igitur ex X_0 (coefficiente anteriore) derivatur, et quidem hunc prævie secundum dx vulgari modo derivando deindeque singulariter ut antea differentiando; quæ quidem regula generalis est: *ut enim* X_{r+1} *ab* X_r *obtineatur, aliud haud opus est, quam*

$(\underset{1}{\delta} X_r =)$ *ipsum* X_r *jam explicitum secundum* dx *derivare* (i. e. $s\widehat{dx}^{s-1}$ loco $\widehat{dx}^{s}$ ubique scribere), *quodque ita exoritur, secundum regulam jam traditam* $(\overset{1}{\underset{c}{d}})$ *singulariter differentiare.* Id quod ita significari potest: $X_{r+1}=\overset{1}{\underset{c}{d}}(\underset{1}{\delta} X_r)$. Ita Ex. gr. ex $X_0=\widehat{dx}^{n}$ obtinetur primum $\underset{1}{\delta} X_0=n\widehat{dx}^{n-1}$, & deinde $\overset{1}{\underset{c}{d}}(dx)^{n-1}=\frac{n-1}{2}\widehat{dx}^{n-2}d^2x$, atque igitur $X_1=\overset{1}{\underset{c}{d}}(n.\widehat{dx}^{n-1}) = \frac{n.\overline{n-1}}{2}\widehat{dx}^{n-2}d^2x$, ut jam observavimus. Tum ex $X_1=\frac{n.\overline{n-1}}{2}\widehat{dx}^{n-2}d^2x$ oritur primum $\underset{1}{\delta} X_1=\frac{n.\overline{n-1}}{2}.\overline{n-2}\,\widehat{dx}^{n-3}d^2x$, deindeque

$$X_2=\overset{1}{\underset{c}{d}}(\underset{1}{\delta} X_1)=\frac{n.\overline{n-1}}{2}.\overline{n-2}.\left(\frac{\overline{n-3}}{2.2}.\widehat{dx}^{n-4}\,\widehat{d^2x}^{2}+\widehat{dx}^{n-3}.\frac{d^3x}{3}\right).$$

Ulteriusque $\underset{1}{\delta} X_2=\frac{n.\overline{n-1}}{2}.\overline{n-2}.\Big(\frac{\overline{n-3}}{4}.\overline{n-4}\,\widehat{dx}^{n-5}(d^2x)^2 +$ $+\frac{\overline{n-3}}{3}.\widehat{dx}^{n-4}.d^3x\Big)$, indeque $X_3=\overset{1}{\underset{c}{d}}(\underset{1}{\delta} X_2)=\frac{n.\overline{n-1}}{2}.\overline{n-2}.\overline{n-3}.\frac{\overline{n-4}}{4}.$

$$\left(\frac{\overline{n-5}\,\widehat{dx}^{n-6}\,\widehat{d^2x}^{3}}{2.3}+\widehat{dx}^{n-5}.2d^2x.\frac{d^3x}{3}\right)+\frac{n}{2}.\overline{n-1}.\overline{n-2}.\frac{n-3}{3}.\widehat{dx}^{n-4}.\frac{d^4x}{4}.$$

Eundemque in modum successive calculabis X_4, X_5, X_6 etc.

Positis igitur $X_0=(dx)^n$, (et n_r (ut $n_2=\frac{n.\overline{n-1}}{2}$) = coefficiente binomii erit $X_1=n_2(dx)^{n-2}d^2x$ seu breviter

$$X_1=\overset{1}{\underset{c}{d}}(\underset{1}{\delta} X_0)=n_2\widehat{dx}^{n-2}d^2x$$

$$X_2=n_3\widehat{dx}^{n-3}d^3x+3.n_4\widehat{dx}^{n-4}(d^2x)^2=\overset{1}{\underset{c}{d}}(\underset{1}{\delta} X_1)=(\underset{c}{d}\underset{1}{\delta})^2X_0,$$

$X_3 = n_4 \widehat{dx}^{n-4} d^4x + 10 n_5 \widehat{dx}^{n-5} d^2x d^3x + 15 n_6 \widehat{dx}^{n-6} (d^2x)^3 = \overset{1}{\underset{c}{d}}(\partial X_2)$,

$X_4 = n_5 \widehat{dx}^{n-5} d^5x + 10 n_6 \widehat{dx}^{n-6} \widehat{d^3x}^2 + 15 n_6 \widehat{dx}^{n-6} d^2x d^4x + 105 n_7 . \widehat{dx}^{n-7} \widehat{d^2x}^2 d^3x +$

$+ 105 . n_8 \widehat{dx}^{n-8} \widehat{d^2x}^4 = \overset{1}{\underset{c}{d}}(\underset{1}{\partial} X_3)$, etc., generatim $X_r = (\overset{1}{\underset{c}{d}} \underset{1}{\partial})^r X_0$, seu

$X_1 = n_2 \widehat{dx}^{n-2} d^2x$

$X_2 = n_3 dx^{n-4} . \left(dx . d^3x + 3 . \frac{n-3}{4} . \widehat{d^2x}^2 \right)$.

$X_3 = n_4 dx^{n-6} . \left(\widehat{dx}^2 d^4x + 10 . \frac{n-4}{5} . dx d^2x d^3x + 15 . \frac{n-4}{5} . \frac{n-5}{6} . \widehat{d^2x}^3 \right)$

generatimque $X_{r+1} = \overset{1}{\underset{c}{d}}(\underset{1}{\partial} X_r)$. Hinc igitur habetur formula generalis ipsius $d^n f = X_0 \partial^n f + X_1 \partial^{n-1} f + X_2 \partial^{n-2} f + X_3 \partial^{n-3} f + \ldots + X_{n-2} \partial^2 f + X_{n-1} \partial f$, seu $d^n f = \widehat{dx}^n . \partial^n f + n_2 dx^{n-2} d^2x \partial^{n-1} f + (n_3 \widehat{dx}^{n-3} d^3x + 3 n_4 \widehat{dx}^{n-4} (d^2x)^2) \partial^{n-2} f +$ &c. $+ d^n x . \partial f$, quæ differentiale cujusvis ordinis breviter exprimit.

Ex. gr. Ponatur $n = 6$, eritque

$d^6 f = \widehat{dx}^6 \partial^6 f + 15 \widehat{dx}^4 d^2x . \partial^5 f + (45 \widehat{dx}^2 \widehat{d^2x}^2 + 20 dx^3 d^3x) \partial^4 f +$

$+ (15 \widehat{d^2x}^3 + 60 dx d^2x d^3x + 15 \widehat{dx}^2 d^4x) \partial^3 f +$

$+ (10 \widehat{d^3x}^2 + 15 d^2x d^4x + 6 dx d^5x) \partial^2 f + d^6x . \partial f$;

quod etiam differentiale ex regulis antea traditis, ipsum $d^5 f$ jam exhibitum singulari nostro modo differentiando, facile obtinebis.

8) Neque silentio prætereundus est **alius modus**, hos coefficientes vere differentiales X_0, X_1, X_2, ... X_{n-1}, X_n definiendi, idque tanto minus, cum ipse ab ultimis X_n, X_{n-1} aptissime initium faciat. Hic modus scilicet seorsim differentialium complexus seorsimque horum coefficientes efformat. Ipsi vero complexus in $d^n f$ præter derivata functionis f differen-

tialia habent ipsius x, quorum exponentes, interni cum externis, ex *combinatione ad summam* n oriuntur. Sic Ex. gr. pro $n=6$, complexus combinationum ad summam 6, sunt $^{1})$ 6; $^{2})$ 15, 24, 33; $^{3})$ 114, 123, 222; $^{4})$ 1113, 1122; $^{5})$ 11112; $^{6})$ 111111; indeque hi differentialium oriuntur acervi (cum $d^1x=dx$ sit): 1) d^6x; 2) dxd^5x, d^2xd^4x, $\widehat{d^3x}^2$; 3) $\widehat{dx}^2 d^4x$, dxd^2xd^3x, $\widehat{d^2x}^3$; 4) $\widehat{dx}^3d^3x$, $\widehat{dx}^2\widehat{d^2x}^2$; 5) $(dx)^4d^2x$; 6) $(dx)^6$; quorum diversi ordines (seu *classes* ut vulgo audiunt) ad suam quique ipsius f functionem derivatam cum numero justo N juncti, totum d^6f constituent: nempe

$$d^6f=d^6x.\partial f+(N\widehat{1,5}.dxd^5x+N\widehat{2,4}.d^2x.d^4x+N\widehat{3^2}.\widehat{d^3x}^2)\partial^2 f+$$
$$+(N\widehat{1^2,4}.\widehat{dx}^2d^4x+N\widehat{1,2,3}.dxd^2xd^3x+N2^3.\widehat{d^2x}^3)\partial^3 f+$$
$$+(N\widehat{1^3,3}.\widehat{dx}^3d^3x+N\widehat{1^2,2^2}.\widehat{dx}^2d^2x)\partial^4 f+$$
$$+N\widehat{1^4,2}\widehat{dx}^4d^2x\partial^5 f+\widehat{dx}^6\partial^6 f.$$

Ejusmodi vero complexus ex regulis notissimis doctrinæ combinationum efformantur. (Ex. gr. ${}^1C^n=n$; ${}^2C^n=1\,\widehat{n-1}$, $2\,\widehat{n-2}$, ... $r\,\widehat{n-r}$ .... seu ${}^2C^n=1.{}^1C^{n-1}+2.{}^1C^{n-2}+3.{}^1C^{n-3}+\ldots$ (= classis secunda combinationum ad summam n); ${}^3C^n=1.{}^2C^{n-1}+2.{}^2C^{n-2}+3.{}^2C^{n-3}+\ldots$ seu ${}^3C^n=11\,\widehat{n-2}$, $12\,\widehat{n-3}$, $13\,\widehat{n-4}$ ....; $22\,\widehat{n-4}$, $23\,\widehat{n-5}$, etc. .....).

Quod vero ad numeros attinet, hi tum diverso modo ex sese tum absolute formantur. Sit nempe

$N(1^\alpha,2^\beta,3^\gamma\ldots r^\varrho).\widehat{dx}^\alpha.\widehat{d^2x}^\beta.\widehat{d^3x}^\gamma\ldots.\widehat{d^rx}^\varrho\partial^{n-r}f$ terminus aliquis in d^nf, similisque

$M(1^{\alpha'},2^{\beta'},3^{\gamma'},\ldots r^{\varrho'},)\widehat{dx}^{\alpha'}.\widehat{d^2x}^{\beta'}.\widehat{d^3x}^{\gamma'}\ldots.\widehat{d^rx}^{\varrho'}.\partial^{n-r'}f$ in $d^{n-1}f$;

Eritque, cum $d^nf=n\overset{1}{\underset{c}{d}}(d^{n-1}f)$ sit,

$$n.\overset{1}{\underset{c}{d}}(M\overbrace{1^\alpha,2^\beta,3^\gamma,\ldots.\overbrace{r-2}^{\nu},}r^{\varrho'}.\widehat{dx}^\alpha\widehat{d^2x}^\beta\widehat{d^3x}^\gamma\ldots.\overbrace{d^{r-2}x}^{\nu}.\widehat{d^rx}^{\varrho'})\partial^{n-r}f=$$

$=n.M\overline{1^{\alpha},2^{\beta},3^{\nu}\ldots\overline{r-2}^{\nu},r^{\varrho'}}.\overline{dx^{\alpha}}.\overline{d^2x^{\beta}}\overline{d^3x^{\gamma}}\ldots\overline{d^{r-2}x^{\nu}}.\varrho'\overline{d^rx}^{\varrho'-1}\frac{d^{r+1}x}{r+1}.\delta^{n-r}f$ æquale termino correspondenti in d^nf, ideoque

$$N\overline{1^{\alpha},2^{\beta},3^{\gamma}\ldots\overline{r-2}^{\nu},r^{\varrho'-1},r+1}=\frac{n\varrho'}{r+1}.M\overline{1^{\alpha},2^{\beta},3^{\gamma}\ldots\overline{r-2}^{\nu},r^{\varrho'}}$$

seu $N\overline{1^{\alpha},2^{\beta},3^{\gamma}\ldots\overline{r-2}^{\nu},r^{\varrho},r+1}=n.\frac{\varrho+1}{r+1}.M\overline{1^{\alpha},2^{\beta},3^{\gamma},\ldots\overline{r-2}^{\nu},r^{\varrho+1}}$

ideoque numerus N(in d^nf seu ad exponentem n pertinens) hujus formulæ ope ex coefficiente M, ad exponentem $(m=)$ $n-1$ pertinente, computatur. Sit vero $\alpha+2\beta+3\gamma+\ldots+\overline{r-2}\,\nu+r\varrho+r+1=n$, qui numerus combinationum constituebat summam.

Demum observandum est, coefficientem $N\overline{1^{\alpha}2^{\beta}3^{\gamma}\ldots\ldots}$ absolute hunc in modum exprimi:

$$N\overline{1^{\alpha}2^{\beta}3^{\gamma}\ldots\ldots}=\frac{\Gamma\overline{\alpha+2\beta+3\gamma+\ldots}}{\Gamma\alpha.\Gamma\beta.\Gamma\gamma\ldots\Gamma1^{\alpha}.\Gamma2^{\beta}.\Gamma3^{\gamma}\ldots}\,,\quad \text{existente}\quad \Gamma x=1.2.3\,\ldots\ldots x,$$

$(\Gamma1=1=\Gamma o)$, atque $\alpha+2\beta+3\gamma+\ldots=n$. Ex. gr. $N(3^2)=\frac{\Gamma6}{\Gamma2.\Gamma3^2}=\frac{4.5}{2}=10$.

Cum igitur ${}^1C^n=n$, ${}^2C^n=1\,\overline{n-1},2\,\overline{n-2},3\,\overline{n-3},\ldots{}^3C^n=1\,{}^2C^{n-1}+2\,{}^2C^{n-2}+\ldots$ sit, atque symbolice $X_n=(\delta)\,{}^1C^n(x)$, $X_{n-r}^{\cdots}=(\delta)\,{}^2C^n\,(x)$, etc.; erit

$$d^nf=d^nx.\delta f+(N\overline{1^1\overline{n-1}^1}dxd^{n-1}x+N\overline{2^1\overline{n-2}^1}d^2xd^{n-2}x+\ldots)\delta^2f+$$

$$+[N\overline{1^2\overline{n-2}}.dx^2.d^{n-2}x+N\overline{1^12^1\overline{n-3}^1}.dxd^2xd^{n-3}x+\ldots$$

$$N\overline{1^1\,3^1\,\overline{n-4}}\,dxd^3xd^{n-4}x+N\overline{2^2\,\overline{n-4}}.\overline{d^2x}^2d^{n-4}x+$$

$$+N\overline{2^13^1\overline{n-5}^1}\,d^2xd^3xd^{n-5}x+\ldots\ldots)\;\delta^3f+\ldots\quad\text{seu}$$

$$d^nf=d^nx.\delta f+\left(\frac{\Gamma n}{\Gamma1.\Gamma\overline{n-1}}.dxd^{n-1}x+\frac{\Gamma n}{\Gamma\overline{n-2}.\Gamma2}.d^2xd^{n-2}x+\ldots\right)\delta^2f+\ldots$$

$$+\left(\frac{\Gamma n}{\Gamma 2.\Gamma\overline{n-2}}dx^2d^{n-2}x+\frac{\Gamma n}{\Gamma 2.\Gamma\overline{n-3}}dxd^2xd^{n-3}x+\ldots\right)\partial^3 f+\ldots,\quad \text{seu}$$

$$d^nf=d^nx.\partial f+(ndxd^{n-1}x+n_2d^2xd^{n-2}x+n_3d^3xd^{n-3}x+\ldots.)\ \partial^2 f+$$
$$+(n_2dx^2d^{n-2}x+3n_3dxd^2xd^{n-3}x+\ldots.)\ \partial^3 f+\ldots.$$

His jam regulis effectum est, ut $d^n(fx)$ pro quocunque valore integro ipsius (n) vel immediate in terminis explicitis absque ulla reductione habeatur, vel successive ex differentialium anteriorum valoribus explicitis et immediate reductis evolvatur, ut hac parte parum ulterius desiderari vel superaddi posse videatur. Nostrarum vero regularum, quas juvenes conscripsimus, *demonstratio* ex theoria derivationis ampliori et disciplina combinationum peti potest, heic vero utut satis prolixa, prætermissa est. Simplicius forsan ita hæc res efficiatur, ut primum observetur differentiationem completam, coefficientibus constantibus sepositis, alios haud præbere terminos variabiles, ac quæ differentiatio secundum duo altissima in unoquoque termino præsentia sibique consequentia differentialia suppeditet; deinde eligatur functio specialis (ex. gr. potentia), cujus evolutione coefficientes determinari possint; Id quod aliis juvenibus disquirendum relinquimus. Nec spernendi erunt admininiculi *Hindenburgianæ* de polynomio elevando regulæ; quas igitur interim consulas.

Patet enim has res arcte cohærere. Vidimus enim antea, hanc evolutionem

$\varphi(z+rdz+r^2\underset{c}{d^2}z+\ldots)$, quam per

$\varphi z+rd\varphi z+r^2\underset{c}{d^2}\varphi z+r^3\underset{c}{d^3}\varphi z+\ldots$ indicavimus, ad hanc revocari

$\varphi(z+r\zeta)=\varphi z+r\zeta\varphi_1 z+r^2\zeta^2\varphi_2 z+r^3\zeta^3\varphi_3+\ \ldots\ ,$ ubi

$\zeta=dz+r\underset{\circ}{d^2}z+r^2\underset{\circ}{d^3}z+\ \ldots\ ,$ ut igitur totum negotium ad potentiam

$\zeta^n = (z_1 + r z_2 + r^2 z_3 + \ldots)^n$ rite evolvendam redeat, de qua & Hindenburg egit & nobis seorsim agendi est animus.

Coronidis

Loco *novum & analysin & ipsam functionum evolutionem considerandi modum* breviter indicare lubet.

Jam enim Præs. partim in tractatu Gothoburgi eruditis biennio ante exhibito partim in lectionibus tam publicis quam privatis monstravit, ipsam operationem Matheseos primariam, a qua ceteræ originem ducunt, *additionem* dico, multo generalius concipi posse, ut ea sit, quæ hac gaudeat proprietate, $(x|y)|z = x|y|z = (z|y)|x$, exsistentibus x, y & z quibusvis, prætereaque $x|\mathfrak{Z} = x$, exsistente $\mathfrak{Z}$ determinato quodam, huic operationi *indifferente*. Talia vero *aggregata* $x|y$ innumera dantur, ut $x+y+axy$, $\frac{x+y+axy}{1-cxy}$ &c. Viceversa vero si $x|y=z$, erit x *differentia formalis* (si $z \lfloor\overline{y}$ vel $z|\overline{y}$ vel $z \lceil y$) ipsorum z & y.

Quibus positis, *producta generaliora* $x|x = 2 \odot x$, $x|x|x = 3 \odot x$, $x|x|x| \ldots |x = n \odot x$, seu $(n+1) \odot x = (n \odot x)|x$ per aggregata illa explicantur, proprietatesque ipsis producti communis simillimas his vindicantur, idque eodem fere modo, ac quo *doctrina proportionum* stabilitur:

$$\text{Ut } n \odot (a|b) = \overparen{n \odot a}\,|\,\overparen{n \odot b}$$

$$n \odot (r \odot a) = r \odot (n \odot a) = (n.r) \odot a$$

$$n \odot (a \lceil b) = \overparen{n \odot a}\,\lceil\,\overparen{n \odot b}$$

$$\overbrace{n \odot a} \,|\, \overbrace{r \odot a} = (n+r) \odot a$$

$$\overbrace{n \odot a} \,\Gamma\, \overbrace{r \odot a} = \overbrace{n - r} \odot a.$$

Et si $x \odot y = z$ fuerit, erit invertendo $x = z\, \overline{\odot\, y}$ vel $= z \odot^{-} y$ *Hyperlogarithmus* seu *exponens generalis* ipsorum z & y, atque $y = \overline{x \odot}\, z$ *Antilogarithmus functionalis* ipsorum x & z. De quibus formulæ istis correspondentes valent. Jam vero, quia aggregatum hoc generale $x \,|\, y$ eadem gaudet proprietate $x \,|\, y \,|\, z = (x \,|\, y) \,|\, z = x \,|\, (y \,|\, z)$, qua demonstratio Theorematis Taylorei nititur (ut p. 4 sq. vidisti); eodem fere modo simile, sed multo latius patens *Theorema* demonstratur, quod ita sonat:

$F(x \,|\, y) = Fx \,|\, (\varLambda y \odot \mathfrak{D} Fx) \,|\, \widehat{\varLambda y}^2 \odot \underset{c}{\mathfrak{D}}^2 Fx \,|\, \widehat{\varLambda y}^3 \odot \underset{c}{\mathfrak{D}}^3 Fx \,|\, ..$, exsistente $\varLambda y$ certa ad ipsum aggregatum ($|$) ita relata functione, ut sit $\varLambda(x \,|\, y) = \varLambda x + \varLambda y$, (ideoque ipsius hyperlogarithmus), atque $\mathfrak{D} Fx$ functione hoc ipso novo maximoque generali evolvendi modo ex functione F (cum $\varLambda$) *derivata*; ceteris vero derivatis $\underset{c}{\mathfrak{D}}^2 Fx$, $\underset{c}{\mathfrak{D}}^3 Fx$ &c. ex prima ita generatis, ut sit

$$\underset{c}{\mathfrak{D}}^2 Fx = \tfrac{1}{2} \odot \mathfrak{D}(\mathfrak{D} Fx) = \tfrac{1}{2} \odot \mathfrak{D}^2 Fx,\ \underset{c}{\mathfrak{D}}^3 Fx = \tfrac{1}{2} \odot \tfrac{1}{3} \odot \mathfrak{D}\mathfrak{D}^2 Fx = \tfrac{1}{2.3} \odot \mathfrak{D}^3 Fx,$$

universimque $\underset{c}{\mathfrak{D}}^n Fx = \dfrac{1}{\Gamma n} \odot \mathfrak{D}^n Fx$, exsistente $\Gamma n = 1.2.3 \ldots n = n.\Gamma\overbrace{n-1}$.

Speciatim igitur si $x \,|\, y = x + y$ posueris, ipsum Theorema Tayloreum inde facile colliges. Sin vero $x \,|\, y = x + y + axy$ & $\varLambda x = L(1 + ax)$ posueris, aliud Theorema tibi constabit, similiterque innumera isti æmula colliges. Tota vero hæc res melius in *Theoria functionum iteratarum* explicatur.